Two remarks on C^∞ Anosov diffeomorphisms

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ABSTRACT. Let M be a closed oriented C^∞ manifold and f a C^∞ Anosov diffeomorphism on M. We show that if M is the two torus T^2, then f is conjugate to a hyperbolic automorphism of T^2, either by a C^∞ diffeomorphism or by a singular homeomorphism. We also show that for general M, if f admits an absolutely continuous invariant measure μ, then μ is a C^∞ volume. The proofs are concatenations of well known results in the field.

1. Conjugacy

Let f be a C^∞ Anosov diffeomorphism on the two torus T^2. Then

$$A = f_* \in \mathrm{Aut}(H_1(T^2, \mathbb{Z})) = \mathrm{SL}(2, \mathbb{Z})$$

defines a hyperbolic automorphism of the abelian Lie group T^2, and f is isotopic to A. It is known [**F, M**] that f is conjugate to A by a homeomorphism h which is isotopic to the identity: $h \circ A = f \circ h$. It is well known that the conjugacy h is a bi-Hölder homeomorphism. Also it is easy to show that h is unique. Let us denote by m the normalized Haar measure of T^2. A homeomorphism h of T^2 is said to be *singular* if there is an m-conull Borel set E such that $h(E)$ is m-null. Our first result is the following.

THEOREM 1. *The conjugacy h is either a C^∞ diffeomorphism or a singular homeomorphism.*

PROOF. Let $TT^2 = E^u \oplus E^s$ be the hyperbolic splitting associated with f. By a dimensional reason, it is a C^1 splitting [**MM1**]. Fix a translation invariant C^∞ Riemannian metric g on T^2. The derivative of f along E^u (resp. E^s) measured with respect to g is denoted by $J^s f$ (resp. $J^u f$). These are C^1 functions. The Gibbs measure [**S, B**] for the potential $-\log|J^u f|$ (resp. $\log|J^s f|$) is denoted by μ_+ (resp. μ_-).

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Let f, A and h be as above. First consider the case where f does not admit an a. c. i. m. (absolutely continuous invariant measure). Then the f-invariant measure h_*m is singular to m.

To show this, notice that h_*m is decomposed into two parts; one absolutely continuous and the other singular. Since h is a C^∞ diffeomorphism, it leaves each part invariant. But the absolutely continuous part must be zero since by the assumption there is no a. c. i. m. for f.

Thus h maps the measure m to a singular measure h_*m. Since h_*m is singular, there is an m-null set E' such that $h_*m(E') = m(h^{-1}E') = 1$. The conjugacy h maps the m-conull set $h^{-1}(E')$ to the m-null set E', showing that h is a singular homeomorphism.

Next consider the case where f admits an a. c. i. m. μ. Then we have $\mu = \mu_+ = \mu_-$ (Proof of Corollary 1 of [**S**], Corollary 4.13 of [**B**]). In particular an a. c. i. m. μ is unique and ergodic. The induced measure h_*m is also ergodic. Therefore either μ and h_*m are mutually singular or coincide. In the former case, we argue just as before, to conclude that the conjugacy h is a singular homeomorphism.

Finally assume that $\mu_+ = \mu_- = h_*m$. These are the Gibbs measures of three potentials, $-\log|J^uf|$, $\log|J^sf|$ and a constant. By Section 3.4 of [**S**], these three functions, with the identical Gibbs measure, are mutually cohomologous modulo constant. That is, there are continuous functions v_1, v_2 and constants c_1, c_2 such that

$$-\log|J^uf| = v_1 \circ f - v_1 + c_1,$$

$$\log|J^sf| = v_2 \circ f - v_2 + c_2.$$

This shows that the Lyapunov exponents of all periodic orbits are the same. By Theorem 1 of [**MM2**], the conjugacy h is a C^∞ diffeomorphism. The proof of Theorem 1 is complete. □

2. Absolutely continuous invariant measure

Let M be a closed oriented n-dimensional C^∞ manifold and f a C^∞ Anosov diffeomorphism on M. Let g be a C^∞ Riemannian metric on M, and m the normalized measure given by the volume form associated with g.

THEOREM 2. *Assume f admits an a. c. i. m. μ with density φ: $\mu = \varphi m$, $\varphi \in L^1(m)$. Then the density φ is a positive C^∞ function.*

PROOF. Let $TM = E^u \oplus E^s$ be the hyperbolic splitting associated with f. Denote the Jacobian along E^u (resp. E^s) measured with respect to g by J^uf (resp. J^sf). The total Jacobian measured with respect to g is denoted by Jf. All these are continuous real valued functions on M.

Define another continuous Riemannian metric g' by $g' = g|_{E^u} \oplus g|_{E^s}$. Thus E^u and E^s are perpendicular with respect to g'. Let m' be the normalized measure given by the volume form associated with g'. We have $m' = e^a m$ for a continuous function a.

Denote by $J'f$ the total Jacobian with respect to g'. Then we have

$$\log|J'f| = \log|J^u f| + \log|J^s f|. \tag{2.1}$$

By [**S, B**], we have $\mu = \mu_+ = \mu_-$, where μ_+ (resp. μ_-) is the Gibbs measure for the potential $-\log|J^u f|$ (resp. $\log|J^s f|$). Then by [**S**], $\log|J^u f| + \log|J^s f|$ is cohomologous to a constant. Thus by (2.1), we have

$$\log|J'f| = b \circ f - b + C. \tag{2.2}$$

for a continuous function b and a constant C.

On the other hand, by the invariance of the a. c. i. m. $\mu = \varphi e^{-a} m'$, we have μ-almost everywhere

$$\log|J'f| = (a - \log\varphi) \circ f - (a - \log\varphi) \tag{2.3}$$

Now by (2.3), we have $\mu(\log|J'f|) = 0$. This implies that $C = 0$ in (2.2). Then (2.2) implies the invariance of the measure $e^{-b}m' = e^{-b+a}m$. Moreover, adding an appropriate constant to b, we may assume that $e^{-b+a}m$ is a probability measure. By the uniqueness of the a. c. i. m., we have $\mu = e^{-b+a}m$. That is, the density of μ is positive and continuous. Then by Corollary 2.1 of [**LMM**], we obtain that e^{-b+a} is a C^∞ function. The proof is complete. $\square$

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